

# Fermion-pairing on a square lattice in extreme magnetic fields

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We consider the Cooper-problem on a two-dimensional, square lattice with a uniform, perpendicular magnetic field. Only rational flux fractions are considered. An extended (real-space) Hubbard model including nearest and next nearest neighbor interactions is transformed to “ $k$ -space”, or more precisely, to the space of eigenfunctions of Harper’s equation, which constitute basis functions of the magnetic translation group for the lattice. A BCS-like truncation of the interaction term is performed. Expanding the interactions in the basis functions of the irreducible representations of the point group  $C_{4v}$  of the square lattice simplify calculations. The numerical results indicate enhanced binding compared to zero magnetic field, and thus re-entrant superconducting pairing at extreme magnetic fields, well beyond the point where the usual semi-classical treatment of the magnetic field breaks down.

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## I. INTRODUCTION

The relationship between an external magnetic field and superconductivity is very important both from a theoretical and a practical point of view. It is well known that superconductivity will disappear above a certain magnetic field due to orbital frustration within a semi-classical treatment of the problem, i.e. the pairing states  $(-\mathbf{k}, \mathbf{k})$  are not good quantum numbers. On the other hand, it was realized some years ago that there may be additional interesting effects as a magnetic field is increased to large values at low temperatures. Rasolt and Tešanović<sup>1</sup> showed that re-entrant superconductivity may appear for fields above  $H_{c2}$  due to Landau level quantization. Because electrons in Landau levels will not be influenced by orbital frustration in a magnetic field, an increased magnetic field may in fact enhance superconductivity. The re-entrance has no direct relationship with the low field superconductivity, and some materials may even be superconducting *only* in high magnetic fields! When Zeeman splitting was incorporated the re-entrant superconducting phase disappeared at high enough fields. We will investigate if a similar phenomenon also will happen when a periodic lattice is incorporated, modifying the Landau level picture of the single-particle spectrum to Hofstadter bands.<sup>2</sup> The influence of Zeeman splitting (i.e. effective Landé  $g$ -factor) on the Cooper binding energy, will also be considered.

In order to do this, we have considered the Cooper-problem on a two-dimensional, square lattice with a uniform perpendicular magnetic field, i.e. we consider pairing of fermions in large magnetic fields at zero temperature. The same problem in zero field has been studied in Ref. 3. The model has on-site, nearest neighbor, and next-nearest neighbor interactions, where the values used for the interactions are *input* to the model. We do not have in mind any particular microscopic mechanism caus-

ing these effective interactions, but assume that they can be attractive. Since the interaction potential has finite range, it is possible to expand the Fourier transformed version in a finite number of functions. These are basis functions of the irreducible representations of the point group  $C_{4v}$  of the square lattice.

In the absence of interactions, the Schrödinger equation for the problem is the well known Harper’s equation, whose spectrum and eigenfunctions have been extensively studied, see e.g. Ref. 2,4,5,6,7,8,9,10,11,12,13,14,15,16,17. The spectrum is the so called Hofstadter’s butterfly. If the ratio of the flux per plaquette to the flux quantum is a rational number ( $\frac{\phi}{\phi_0} = \frac{p}{q}$ ), the energy bands split into  $q$  sub-bands. For irrational ratios the spectrum is a Cantor set.<sup>2</sup> Effects of electron correlations on the Hofstadter spectrum have been studied by several authors, see e.g. Ref. 18,19,20,21,22,23, however they do not consider the possibility of Cooper pairing through an attractive effective interaction.

For low magnetic fields the behavior of the system can be estimated by treating the field in a perturbative manner. Using the zero field correlation length one can find a value for  $H_{c2}$ . The limitations of this approach is that it neglects the change in the wave-functions of the Cooper pairs due to the magnetic field. For very large magnetic fields, a more appropriate approach to the problem is to solve the single-particle spectrum first in the presence of the magnetic field, and then introduce the pairing interaction as a singular perturbation on this problem. If one is primarily interested in the behavior near  $T_c(H)$ , one can exploit the fact that the order parameter/energy gap is small and develop linearized formulas as done in e.g. Ref. 24,25,26.

We transform the extended Hubbard model (real-space) to the space of eigenfunctions of the Harper’s equation which diagonalize the single electron part. A BCS-like truncation is then performed and the interac-

tions are expanded in the basis functions of the point group  $C_{4v}$  of the square lattice. From this Hamiltonian, an exact solution of the Cooper problem is in principle possible to obtain, however one encounters the problem that the associated matrix problem increases dramatically upon a reduction of the magnetic field. In fact, the matrix dimensions will be  $\sim q^2 \times q^2$  and only fields far above experimentally achievable ones can be handled in explicit numerical solutions at present.<sup>29</sup> We will consider extremely high fields where the flux per plaquette is a sizeable fraction of  $\phi_0$ , typically  $q < 8$ .

## II. THEORY

### A. The model

We use an extended Hubbard model with nearest and next-nearest neighbor hopping matrix elements  $t$  and  $t'$  respectively, given by

$$H = H_0 + H_I, \quad (1a)$$

where

$$H_0 = \sum_{i,\sigma} [\epsilon(\sigma) - \mu] c_{i,\sigma}^\dagger c_{i,\sigma} - t \sum_{\langle i,j \rangle, \sigma} e^{\phi_{ij}} c_{i,\sigma}^\dagger c_{j,\sigma} - t' \sum_{\langle\langle i,j \rangle\rangle, \sigma} e^{\phi_{ij}} c_{i,\sigma}^\dagger c_{j,\sigma} \quad (1b)$$

and

$$H_I = \frac{U}{2} \sum_{i,\sigma} c_{i,\sigma}^\dagger c_{i,\sigma} c_{i,-\sigma}^\dagger c_{i,-\sigma} + \frac{V}{2} \sum_{\langle i,j \rangle, \sigma, \sigma'} c_{i,\sigma}^\dagger c_{i,\sigma} c_{j,\sigma'}^\dagger c_{j,\sigma'} + \frac{W}{2} \sum_{\langle\langle i,j \rangle\rangle, \sigma, \sigma'} c_{i,\sigma}^\dagger c_{i,\sigma} c_{j,\sigma'}^\dagger c_{j,\sigma'}. \quad (1c)$$

Here  $U$ ,  $V$  and  $W$  are effective interactions between electrons at the same site, nearest neighbor sites and next nearest neighbor sites, respectively. The noninteracting model including next nearest neighbor hopping has been studied in e.g. Ref. 27. The Zeeman splitting is given by  $\epsilon(\sigma) = -g_s \mu_B B \sigma / 2$  where  $\sigma = +(-)[\uparrow(\downarrow)]$ ,  $g_s$  is Landé  $g$ -factor, and  $\mu$  is the chemical potential.<sup>30</sup> The Peierls phase factors<sup>28</sup> for hopping from lattice site  $j$  to  $i$  is

$$\phi_{ij} = -\frac{2\pi}{\phi_0} \int_j^i \mathbf{A} \cdot d\boldsymbol{\ell} \quad (2)$$

where  $\phi_0 = h/e$ . Employing a Landau-gauge  $\mathbf{A} = Bx\hat{e}_y$ , introducing  $g = 2\pi p/q = 2\pi Ba^2/\phi_0$ , and setting  $a = 1$ , the phase factor can be written as

$$\phi_{ij} = \begin{cases} 0 & \text{if } |\mathbf{r}_i - \mathbf{r}_j| = m\hat{e}_x \\ g \frac{x_i + x_j}{2} n & \text{if } |\mathbf{r}_i - \mathbf{r}_j| = m\hat{e}_x + n\hat{e}_y. \end{cases} \quad (3)$$

Using this, and writing the wave function as (we have chosen the normalization  $\sum_{i=1}^N |u_{\mu,\nu,\ell}(x_i)|^2 = 1$  where  $N$  is the number of lattice sites.)

$$\psi_{\mu,\nu,\ell}(x_m, y_m) = \langle \mathbf{r}_m | \boldsymbol{\kappa}, \ell \rangle = e^{i(\mu x_m + \nu y_m)} u_{\mu,\nu,\ell}(x_m), \quad (4)$$

where we have introduced  $\boldsymbol{\kappa} = (\mu, \nu)$ , the Schrödinger equation can be written as

$$\begin{aligned} & -e^{i\mu} \{t + 2t' \cos[g(m+1/2) + \nu]\} u_{\mu,\nu,\ell}(x_{m+1}) \\ & -e^{-i\mu} \{t + 2t' \cos[g(m-1/2) + \nu]\} u_{\mu,\nu,\ell}(x_{m-1}) \\ & - \{2t \cos[gm + \nu]\} u_{\mu,\nu,\ell}(x_m) = \epsilon_{\mu,\nu,\ell} u_{\mu,\nu,\ell}(x_m). \end{aligned} \quad (5)$$

Here,  $x$  is taken modulo  $q$  due to the periodicity for  $m \rightarrow m + q$ , such that the functions  $u$  constitute the periodic part of Bloch functions on the *magnetic* lattice. The equation can be written as a  $q \times q$  matrix. It is furthermore easily seen that the equation is periodic for  $\nu \rightarrow \nu + 2\pi$ . By introducing  $\hat{u}_{\mu,\nu,\ell}(x_m) = e^{i\mu x_m \bmod q} u_{\mu,\nu,\ell}(x_m)$  it is easily shown that the equation for  $\hat{u}_{\mu,\nu,\ell}(x_m)$  is periodic for  $\mu \rightarrow \mu + 2\pi/q$ . As a consequence of the fact that  $x_m - x_{m \bmod q} = nq$ , it is clear that  $\psi_{\mu,\nu,\ell}(x_m, y_m)$  must have the same periodicity. The different eigenvalues are numbered with  $\ell$  and each value corresponds to a different Harper- or Hofstadter-band. Using the completeness relation  $1 = \sum_{\boldsymbol{\kappa}, \ell} |\boldsymbol{\kappa}, \ell\rangle \langle \boldsymbol{\kappa}, \ell|$ , the creation/annihilation operators are seen to transform as<sup>31</sup>

$$\begin{aligned} c_{i,\sigma}^\dagger &= \sum_{\ell=1}^q \sum_{\boldsymbol{\kappa}} e^{-i\boldsymbol{\kappa} \cdot \mathbf{r}_i} u_{\boldsymbol{\kappa}, \ell}^*(x_i) c_{\boldsymbol{\kappa}, \ell, \sigma}^\dagger \\ c_{i,\sigma} &= \sum_{\ell=1}^q \sum_{\boldsymbol{\kappa}} e^{i\boldsymbol{\kappa} \cdot \mathbf{r}_i} u_{\boldsymbol{\kappa}, \ell}(x_i) c_{\boldsymbol{\kappa}, \ell, \sigma} \end{aligned} \quad (6)$$

where  $x_i = x_i \bmod q$ . Inserting Eq. 6 in Eq. 1b we get

$$H_0 = \sum_{\boldsymbol{\kappa}, \ell, \sigma} [\epsilon_{\boldsymbol{\kappa}, \ell} + \epsilon(\sigma) - \mu] c_{\boldsymbol{\kappa}, \ell, \sigma}^\dagger c_{\boldsymbol{\kappa}, \ell, \sigma}. \quad (7)$$

The two-particle terms in Eq. 1c are all of the form

$$\frac{1}{2} \sum_{i,j,\sigma,\sigma'} V(|i-j|) c_{i,\sigma}^\dagger c_{i,\sigma} c_{j,\sigma'}^\dagger c_{j,\sigma'}. \quad (8)$$

By using Eq. 6 and a form of BCS-truncation of the interaction term (See appendix A for details)

$$\begin{aligned} \sigma' &= -\sigma \\ \boldsymbol{\kappa}' &= -\boldsymbol{\kappa}, \quad (-\pi/q, -\pi) \leq \boldsymbol{\kappa} < (\pi/q, \pi), \end{aligned} \quad (9)$$

the two-particle terms can be transformed into

$$\frac{1}{2} \sum_{\ell_1, \ell_2, \ell'_1, \ell'_2} \sum_{\boldsymbol{\kappa}, \boldsymbol{\kappa}', \sigma} V_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'}^{\ell_1 \ell_2, \ell'_1 \ell'_2} c_{\boldsymbol{\kappa}, \ell_1, \sigma}^\dagger c_{-\boldsymbol{\kappa}, \ell_2, -\sigma}^\dagger c_{\boldsymbol{\kappa}', \ell'_1, \sigma} c_{-\boldsymbol{\kappa}', \ell'_2, -\sigma}, \quad (10)$$

where

$$V_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'}^{\ell_1 \ell_2, \ell'_1 \ell'_2} = \sum_{\mathbf{G}, \mathbf{G}'} \sum_{\eta=1}^5 \lambda_\eta \tilde{B}_{\eta, \mathbf{G}, \mathbf{G}'}^* (\ell_1, \ell_2, \boldsymbol{\kappa}) \tilde{B}_{\eta, \mathbf{G}, \mathbf{G}'} (\ell'_1, \ell'_2, \boldsymbol{\kappa}'). \quad (11)$$

Here, we have introduced  $\lambda = (U, V, W, V, W)$  and functions  $\tilde{B}$  given by

$$\tilde{B}_{\eta, \mathbf{G}, \mathbf{G}'}(\ell_1, \ell_2, \kappa) = \tilde{u}_{\kappa, \ell_1}(\mathbf{G}) \tilde{u}_{-\kappa, \ell_2}(\mathbf{G}') B_{\eta}(\kappa), \quad (12a)$$

where

$$\begin{aligned} B_1(\kappa) &= \sqrt{N} \\ B_2(\kappa) &= \sqrt{N} [\cos(\kappa_x) + \cos(\kappa_y)] \\ B_3(\kappa) &= \sqrt{2N} \cos(\kappa_x) \cos(\kappa_y) \\ B_4(\kappa) &= \sqrt{N} [\cos(\kappa_x) - \cos(\kappa_y)] \\ B_5(\kappa) &= \sqrt{2N} \sin(\kappa_x) \sin(\kappa_y), \end{aligned} \quad (12b)$$

and

$$\tilde{u}_{\kappa, \ell}(\mathbf{G}) = \frac{1}{\sqrt{q}} \sum_{i=1}^q u_{\kappa, \ell}(x_i) e^{-i\mathbf{G}x_i}. \quad (12c)$$

Here  $\mathbf{G} = (G_x, G_y)$ , where  $G_x = 0, 2\pi/q, \dots, (q-1)2\pi/q$  is a reciprocal lattice vector for the magnetic lattice, and  $G_y = 0$ .

### B. The Cooper-problem

The two-particle Schrödinger equation is given by

$$(H_0 + H_I)|1, 2\rangle = E|1, 2\rangle. \quad (13)$$

Without interactions the Schrödinger equation is

$$\begin{aligned} H_0|\kappa, \ell_1, \sigma; -\kappa, \ell_2, -\sigma\rangle_0 \\ = (\epsilon_{\kappa, \ell_1, \sigma} + \epsilon_{-\kappa, \ell_2, -\sigma})|\kappa, \ell_1, \sigma; -\kappa, \ell_2, -\sigma\rangle_0, \end{aligned} \quad (14)$$

and it is possible to expand the two-particle states in the complete set  $| \rangle_0$ :

$$\begin{aligned} |1, 2\rangle = \sum_{\kappa, \ell_1, \ell_2, \sigma} \theta(\epsilon_{\kappa, \ell_1, \sigma} - \epsilon_F) \theta(\epsilon_{-\kappa, \ell_2, -\sigma} - \epsilon_F) \\ \times a_{\kappa, \sigma}^{\ell_1, \ell_2} |\kappa, \ell_1, \sigma; -\kappa, \ell_2, -\sigma\rangle_0, \end{aligned} \quad (15)$$

where  $\epsilon_F$  is the Fermi-level and  $\theta(x < 0) = 0$ ,  $\theta(x > 0) = 1$ . Inserting this in the Schrödinger equation Eq. 13, multiplying from the left by  ${}_0\langle \kappa', \ell'_1; -\kappa', \ell'_2 |$ , using Eq. 11, and introducing the shorthand notations  $\chi = \{\kappa, \ell_1, \ell_2\}$  and  $\xi = \{\eta, \mathbf{G}, \mathbf{G}'\}$  give

$$\begin{aligned} \sum_{\xi} \tilde{B}_{\xi}^*(\chi') \underbrace{\sum_{\chi > \chi_F} \lambda_{\xi} \tilde{B}_{\xi}(\chi) a_{\chi, \sigma'}}_{\equiv A_{\xi, \sigma'}} \\ = a_{\chi', \sigma'} [E - (\epsilon_{\kappa', \ell'_1, \sigma'} + \epsilon_{-\kappa', \ell'_2, -\sigma'})], \end{aligned} \quad (16)$$

which implies

$$a_{\chi', \sigma'} = \frac{\sum_{\xi} \tilde{B}_{\xi}^*(\chi') A_{\xi, \sigma'}}{E - (\epsilon_{\kappa', \ell'_1, \sigma'} + \epsilon_{-\kappa', \ell'_2, -\sigma'})}, \quad \chi' > \chi_F. \quad (17)$$

Here the notation  $\chi' > \chi_F$  means  $\epsilon_{\kappa', \ell'_1, \sigma'} > \epsilon_F$  and  $\epsilon_{-\kappa', \ell'_2, -\sigma'} > \epsilon_F$ . Inserting the expression for  $a_{\chi', \sigma'}$  into Eq. 13, multiplying by  $\tilde{B}_{\xi'}(\chi')$  and summing over  $\chi' > \chi_F$  gives

$$\sum_{\xi''} \underbrace{\sum_{\xi} \lambda_{\xi} \Gamma_{\xi', \xi} M_{\xi'', \xi}(E, \sigma') A_{\xi'', \sigma'}}_{\equiv D_{\xi', \xi''}(E, \sigma')} = \sum_{\xi} \Gamma_{\xi', \xi} A_{\xi, \sigma'}, \quad (18)$$

where

$$\Gamma_{\xi', \xi} \equiv \sum_{\chi' > \chi_F} \tilde{B}_{\xi'}^*(\chi') \tilde{B}_{\xi}(\chi'), \quad (19)$$

and

$$M_{\xi'', \xi}(E, \sigma') = \sum_{\chi > \chi_F} \frac{\tilde{B}_{\xi''}^*(\chi) \tilde{B}_{\xi}(\chi)}{E - (\epsilon_{\kappa, \ell_1, \sigma'} + \epsilon_{-\kappa, \ell_2, -\sigma'})}. \quad (20)$$

The final equation is

$$\sum_{\xi} D_{\xi', \xi}(E, \sigma') A_{\xi, \sigma'} = \sum_{\xi} \Gamma_{\xi', \xi} A_{\xi, \sigma'}, \quad (21)$$

which has a (nontrivial) solution if and only if  $\det(D(E) - \Gamma) = 0$ . This determines  $E$  and we can find the vector components  $A_{\xi, \sigma'}$ , and thereby the Cooper wave functions in  $(\kappa, \ell)$ -space,  $a_{\kappa, \sigma'}^{\ell_1, \ell_2}$ . The real space wave function is then

$$\begin{aligned} \langle ij|1, 2\rangle = \sum_{\kappa, \ell_1, \ell_2, \sigma} \theta(\epsilon_{\kappa, \ell_1, \sigma} - \epsilon_F) \theta(\epsilon_{-\kappa, \ell_2, -\sigma} - \epsilon_F) \\ \times a_{\kappa, \sigma}^{\ell_1, \ell_2} \underbrace{\langle i|\kappa, \ell_1, \sigma\rangle_0 \langle j|-\kappa, \ell_2, -\sigma\rangle_0}_{= u_{\kappa, \ell_1}(x_i) u_{-\kappa, \ell_2}(x_j) e^{i\kappa \cdot (r_i - r_j)}}, \end{aligned} \quad (22)$$

or in mass-center and relative coordinates  $X = (x_i + x_j)/2$ ,  $\rho = r_i - r_j$ :

$$\begin{aligned} \langle X, \rho|1, 2\rangle = \sum_{\kappa, \ell_1, \ell_2, \sigma} \theta(\epsilon_{\kappa, \ell_1, \sigma} - \epsilon_F) \theta(\epsilon_{-\kappa, \ell_2, -\sigma} - \epsilon_F) \\ \times a_{\kappa, \sigma}^{\ell_1, \ell_2} u_{\kappa, \ell_1}(X + \frac{\rho_x}{2}) u_{-\kappa, \ell_2}(X - \frac{\rho_x}{2}) e^{i\kappa \cdot \rho}. \end{aligned} \quad (23)$$

### III. RESULTS AND DISCUSSION

In the previous section we have developed a framework for computations involving both nearest- and next nearest neighbor hopping and on-site, nearest neighbor and next nearest neighbor electron interactions. As a first step, we have studied explicit numerical solutions for the situation with only on-site *attractive* potential. This corresponds to setting  $U < 0$  and  $V = W = 0$  in the above formulas, i.e. the Hubbard model which in the absence of a magnetic field is expected to exhibit s-wave superconductivity. The next nearest neighbor hopping term  $t'$  is also set to zero.

We have calculated the Cooper-pair binding energy as function of  $-|U|$  for different values of the band filling  $n$ . A typical result for  $n = 0.4$  is shown in Fig. 1. The results show that the binding energy is an increasing function of  $|U|$  both for  $q = 1$  (corresponds to zero field) and for  $q = 2, 3$ . Neglecting spin-splitting, the binding energy is clearly higher for  $q > 1$  and is an increasing function of  $q$  for the extreme fields we are considering. An increase in  $q$  makes the allowed energy bands narrower, and thereby increases the density of states, and hence this result is to be expected. Since the Fermi level normally lies in a band, the density of states at the Fermi level is higher than in zero field.

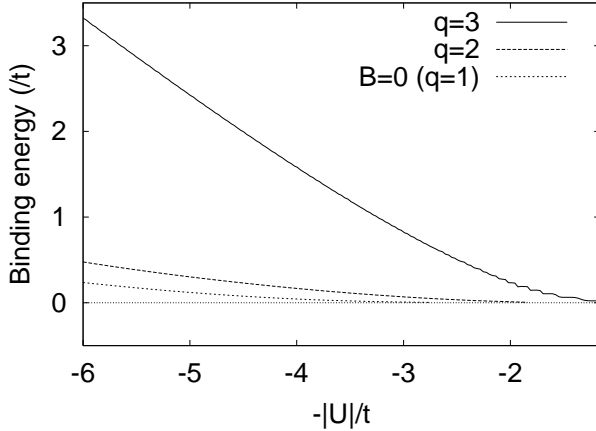


FIG. 1: Binding energy for Cooper pair as function of on-site effective potential for different magnetic fields ( $q$ ). The Zeeman spin splitting is neglected and the band filling is  $n = 0.4$  (0.5 is half filled band).

When one includes the Zeeman spin splitting, the Hofstadter bands will split. As the effective Landé  $g$ -factor increases, the spin up and spin down bands will move relative to each other, and ultimately be completely separated. A typical result for  $U = -4t$  is shown in Fig. 2. The binding energy is not a monotonically decreasing function, but increases and decreases as the bands overlap more and less. When the bands have passed each other and the spin up states are increasingly separated from the spin down states the binding energy is a decreasing function of the  $g$ -factor.

A similar plot as in Fig. 1 is shown in Fig. 3. Here the binding energy as function of  $-|U|$  is plotted for different values of the spin splitting for  $q = 3$ .

#### IV. CONCLUSION

For the attractive Hubbard model in extreme magnetic fields the above results indicate that the binding energy is an increasing function of  $q$ , i.e the binding energy is higher in a magnetic field (extremely strong) than in the zero field situation. It is not clear what will happen for even higher values of  $q$ , but there has to be a maximum

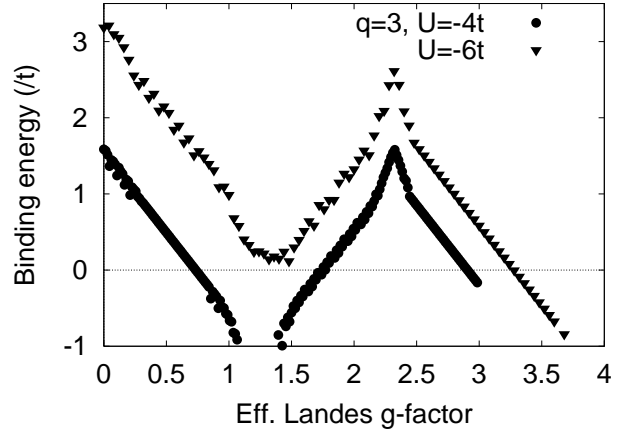


FIG. 2: Binding energy for Cooper pair as function of the effective Landé  $g$ -factor (spin-splitting). The band filling is  $n = 0.4$  and  $q = 3$ .

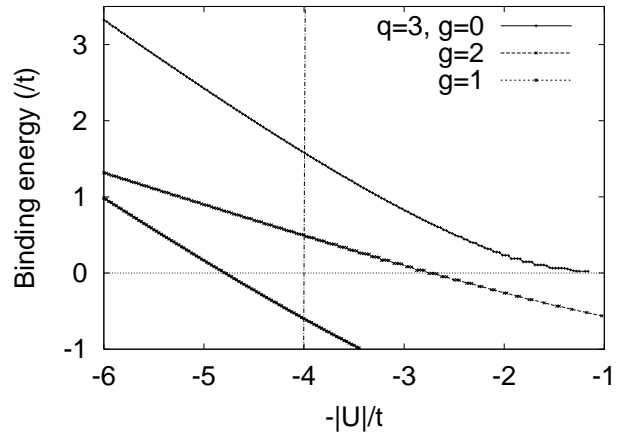


FIG. 3: Binding energy for Cooper pair as function of on-site effective potential for different values of the effective Landé  $g$ -factor for band filling  $n = 0.4$  and  $q = 3$ .

before the decrease towards the normal  $H_{c2}$  from above. If the effective Landé  $g$ -factor is not too high, this result is still valid when we include the Zeeman effect. The results tell us that in the model we have considered, re-entrant superconductivity as a function of magnetic field will appear if the on-site attractive potential is sufficiently strong. For real materials, phase transitions to different phases may appear for lower fields than those we have studied.

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# APPENDIX A: TRANSFORMATION OF THE INTERACTION TERMS

By using Eq. 6 and introducing  $\mathbf{r}_j = \mathbf{r}_i + \boldsymbol{\delta}$ , we can write

$$\begin{aligned}
H_I &= \frac{1}{2} \sum_{i,j,\sigma,\sigma'} V(|i-j|) c_{i,\sigma}^\dagger c_{i,\sigma} c_{j,\sigma'}^\dagger c_{j,\sigma'} \\
&= \frac{1}{2} \sum_{\boldsymbol{\kappa}_1 \dots \boldsymbol{\kappa}_4} \sum_{i,\boldsymbol{\delta}} V(|\boldsymbol{\delta}|) e^{i(\boldsymbol{\kappa}_4 - \boldsymbol{\kappa}_3) \cdot \boldsymbol{\delta}} e^{i[(\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_4) - (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_3)] \cdot \mathbf{r}_i} \\
&\quad \times \sum_{\ell_1 \dots \ell_4} u_{\boldsymbol{\kappa}_1, \ell_1}^*(x_i) u_{\boldsymbol{\kappa}_2, \ell_2}(x_i) u_{\boldsymbol{\kappa}_3, \ell_3}^*(x_i + \delta_x) u_{\boldsymbol{\kappa}_4, \ell_4}(x_i + \delta_x) \\
&\quad \times \sum_{\sigma, \sigma'} c_{\boldsymbol{\kappa}_1, \ell_1, \sigma}^\dagger c_{\boldsymbol{\kappa}_2, \ell_2, \sigma} c_{\boldsymbol{\kappa}_3, \ell_3, \sigma'}^\dagger c_{\boldsymbol{\kappa}_4, \ell_4, \sigma'}.
\end{aligned} \tag{A1}$$

Since  $u_{\boldsymbol{\kappa}, \ell}(x)$  is periodic with period  $q$  in the  $x$ -direction (and 1 in the  $y$ -direction), we can use a Fourier-transform over the *magnetic Brillouin zone*

$$u_{\boldsymbol{\kappa}, \ell}(x_i) = \frac{1}{\sqrt{q}} \sum_{G_x} \tilde{u}_{\boldsymbol{\kappa}, \ell}(G_x) e^{i G_x x_i}, \tag{A2}$$

or

$$u_{\boldsymbol{\kappa}, \ell}(\mathbf{r}_i) = \frac{1}{\sqrt{q}} \sum_{\mathbf{G}} \tilde{u}_{\boldsymbol{\kappa}, \ell}(\mathbf{G}) e^{i \mathbf{G} \cdot \mathbf{r}_i}, \tag{A3}$$

where  $G_x = 0, 2\pi/q, \dots, (q-1)2\pi/q$  and  $G_y = 0$ . By using this, we get

$$\begin{aligned}
H_I &= \frac{1}{2q^2} \sum_{\substack{\boldsymbol{\kappa}_1 \dots \boldsymbol{\kappa}_4 \\ \ell_1 \dots \ell_4}} \sum_{\mathbf{G}_1 \dots \mathbf{G}_4} \tilde{u}_{\boldsymbol{\kappa}_1, \ell_1}^*(\mathbf{G}_1) \tilde{u}_{\boldsymbol{\kappa}_2, \ell_2}(\mathbf{G}_2) \times \tilde{u}_{\boldsymbol{\kappa}_3, \ell_3}^*(\mathbf{G}_3) \tilde{u}_{\boldsymbol{\kappa}_4, \ell_4}(\mathbf{G}_4) \sum_{\boldsymbol{\delta}} V(|\boldsymbol{\delta}|) e^{i(\boldsymbol{\kappa}_4 - \boldsymbol{\kappa}_3) \cdot \boldsymbol{\delta}} e^{i(\mathbf{G}_4 - \mathbf{G}_3) \cdot \boldsymbol{\delta}} \\
&\quad \times \underbrace{\sum_i e^{i[(\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_4) - (\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_3)] \cdot \mathbf{r}_i} e^{i[(\mathbf{G}_2 + \mathbf{G}_4) - (\mathbf{G}_1 + \mathbf{G}_3)] \cdot \mathbf{r}_i}}_{\equiv N \delta_{(\boldsymbol{\kappa}_2 + \mathbf{G}_2) + (\boldsymbol{\kappa}_4 + \mathbf{G}_4), (\boldsymbol{\kappa}_1 + \mathbf{G}_1) + (\boldsymbol{\kappa}_3 + \mathbf{G}_3)}} \sum_{\sigma, \sigma'} c_{\boldsymbol{\kappa}_1, \ell_1, \sigma}^\dagger c_{\boldsymbol{\kappa}_2, \ell_2, \sigma} c_{\boldsymbol{\kappa}_3, \ell_3, \sigma'}^\dagger c_{\boldsymbol{\kappa}_4, \ell_4, \sigma'}.
\end{aligned} \tag{A4}$$

If we then define  $\boldsymbol{\kappa}_q$  by

$$(\boldsymbol{\kappa}_1 + \mathbf{G}_1) - (\boldsymbol{\kappa}_2 + \mathbf{G}_2) = (\boldsymbol{\kappa}_4 + \mathbf{G}_4) - (\boldsymbol{\kappa}_3 + \mathbf{G}_3) \equiv \boldsymbol{\kappa}_q, \tag{A5}$$

and

$$\tilde{V}(\boldsymbol{\kappa}_q) \equiv N \sum_{\boldsymbol{\delta}} V(|\boldsymbol{\delta}|) e^{i \boldsymbol{\kappa}_q \cdot \boldsymbol{\delta}}, \tag{A6}$$

we can write

$$\begin{aligned}
H_I &= \frac{1}{2q^2} \sum_{\substack{\boldsymbol{\kappa}_2, \boldsymbol{\kappa}_3, \boldsymbol{\kappa}_q \\ \ell_1 \dots \ell_4}} \tilde{V}(\boldsymbol{\kappa}_q) \sum_{\mathbf{G}_1 \dots \mathbf{G}_4} \tilde{u}_{\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_q - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1}^*(\mathbf{G}_1) \tilde{u}_{\boldsymbol{\kappa}_2, \ell_2}(\mathbf{G}_2) \tilde{u}_{\boldsymbol{\kappa}_3, \ell_3}^*(\mathbf{G}_3) \tilde{u}_{\boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_q - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4}(\mathbf{G}_4) \\
&\quad \times \sum_{\sigma, \sigma'} c_{\boldsymbol{\kappa}_2 + \boldsymbol{\kappa}_q - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1, \sigma}^\dagger c_{\boldsymbol{\kappa}_2, \ell_2, \sigma} c_{\boldsymbol{\kappa}_3, \ell_3, \sigma'}^\dagger c_{\boldsymbol{\kappa}_3 + \boldsymbol{\kappa}_q - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4, \sigma'}.
\end{aligned} \tag{A7}$$

This can be written on a more symmetric form by letting  $\boldsymbol{\kappa}_2 \rightarrow \boldsymbol{\kappa} - \frac{\boldsymbol{\kappa}_q}{2}$ ,  $\boldsymbol{\kappa}_3 \rightarrow \boldsymbol{\kappa}' - \frac{\boldsymbol{\kappa}_q}{2}$ :

$$\begin{aligned}
H_I &= \frac{1}{2q^2} \sum_{\substack{\boldsymbol{\kappa}, \boldsymbol{\kappa}', \boldsymbol{\kappa}_q \\ \ell_1 \dots \ell_4}} \tilde{V}(\boldsymbol{\kappa}_q) \sum_{\mathbf{G}_1 \dots \mathbf{G}_4} \tilde{u}_{\boldsymbol{\kappa} + \frac{\boldsymbol{\kappa}_q}{2} - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1}^*(\mathbf{G}_1) \tilde{u}_{\boldsymbol{\kappa} - \frac{\boldsymbol{\kappa}_q}{2}, \ell_2}(\mathbf{G}_2) \tilde{u}_{\boldsymbol{\kappa}' - \frac{\boldsymbol{\kappa}_q}{2}, \ell_3}^*(\mathbf{G}_3) \tilde{u}_{\boldsymbol{\kappa}' + \frac{\boldsymbol{\kappa}_q}{2} - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4}(\mathbf{G}_4) \\
&\quad \times \sum_{\sigma, \sigma'} c_{\boldsymbol{\kappa} + \frac{\boldsymbol{\kappa}_q}{2} - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1, \sigma}^\dagger c_{\boldsymbol{\kappa} - \frac{\boldsymbol{\kappa}_q}{2}, \ell_2, \sigma} c_{\boldsymbol{\kappa}' - \frac{\boldsymbol{\kappa}_q}{2}, \ell_3, \sigma'}^\dagger c_{\boldsymbol{\kappa}' + \frac{\boldsymbol{\kappa}_q}{2} - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4, \sigma'}.
\end{aligned} \tag{A8}$$

Now we perform a BCS like truncation

$$\begin{aligned}\sigma' &= -\sigma \\ \kappa' &= -\kappa; \quad (-\pi/q, -\pi) \leq \kappa < (\pi/q, \pi),\end{aligned}\tag{A9}$$

where  $\kappa$  is in the magnetic Brillouin zone, i.e. we assume that the Cooper pair is a spin singlet. Due to the antisymmetry of the spin part of the wave-function, the space part has to be symmetric. If  $\kappa' + \kappa \neq 0$ , the wave-function for the pairs center of mass would have a modulation. This modulation will generally be incommensurable with the underlying lattice. Thus

$$\begin{aligned}H_I &= \frac{1}{2q^2} \sum_{\substack{\kappa, \kappa_q, \sigma \\ \ell_1 \dots \ell_4}} \tilde{V}(\kappa_q) \sum_{\mathbf{G}_1 \dots \mathbf{G}_4} \tilde{u}_{\kappa + \frac{\kappa_q}{2} - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1}^* (\mathbf{G}_1) \tilde{u}_{\kappa - \frac{\kappa_q}{2}, \ell_2} (\mathbf{G}_2) \tilde{u}_{-(\kappa + \frac{\kappa_q}{2}), \ell_3}^* (\mathbf{G}_3) \tilde{u}_{-(\kappa - \frac{\kappa_q}{2}) - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4} (\mathbf{G}_4) \\ &\quad \times c_{\kappa + \frac{\kappa_q}{2} - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1, \sigma}^\dagger c_{\kappa - \frac{\kappa_q}{2}, \ell_2, \sigma} c_{-(\kappa + \frac{\kappa_q}{2}), \ell_3, -\sigma}^\dagger c_{-(\kappa - \frac{\kappa_q}{2}) - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4, -\sigma}.\end{aligned}\tag{A10}$$

By letting  $\kappa + \frac{\kappa_q}{2} \rightarrow \kappa$ , and  $\kappa - \frac{\kappa_q}{2} \rightarrow \kappa'$  we get

$$\begin{aligned}H_I &= \frac{1}{q^2} \sum_{\substack{\kappa, \kappa', \sigma \\ \ell_1 \dots \ell_4}} \tilde{V}(\kappa - \kappa') \sum_{\mathbf{G}_1 \dots \mathbf{G}_4} \tilde{u}_{\kappa - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1}^* (\mathbf{G}_1) \tilde{u}_{\kappa', \ell_2} (\mathbf{G}_2) \tilde{u}_{-\kappa, \ell_3}^* (\mathbf{G}_3) \tilde{u}_{-\kappa' - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4} (\mathbf{G}_4) \\ &\quad \times c_{\kappa - (\mathbf{G}_1 - \mathbf{G}_2), \ell_1, \sigma}^\dagger c_{\kappa', \ell_2, \sigma} c_{-\kappa, \ell_3, -\sigma}^\dagger c_{-\kappa' - (\mathbf{G}_4 - \mathbf{G}_3), \ell_4, -\sigma}.\end{aligned}\tag{A11}$$

To transform this back to the magnetic Brillouin zone we use Eq. 12c, in addition to the fact that the combination  $e^{-i\kappa \cdot \mathbf{r}_i} u_{\kappa, \ell}^*(x_i) c_{\kappa, \ell, \sigma}^\dagger$  has to be invariant if we let  $\kappa \rightarrow \kappa + \mathbf{G}$  (as can be seen from Eq. 6). Then

$$u_{\kappa + \mathbf{G}, \ell}^*(x_i) c_{\kappa + \mathbf{G}, \ell, \sigma}^\dagger = e^{-i\mathbf{G} \cdot \mathbf{r}_i} u_{\kappa, \ell}^*(x_i) c_{\kappa, \ell, \sigma}^\dagger.\tag{A12}$$

We can then write Eq. A11 as

$$H_I = \frac{1}{2q^2} \sum_{\substack{\kappa, \kappa', \sigma \\ \ell_1 \dots \ell_4}} \tilde{V}(\kappa - \kappa') \frac{1}{q^2} \sum_{\substack{\mathbf{G}_1 \dots \mathbf{G}_4 \\ \mathbf{r}_1 \dots \mathbf{r}_4}} u_{\kappa, \ell_1}^*(\mathbf{r}_1) u_{\kappa', \ell_2}(\mathbf{r}_2) u_{-\kappa, \ell_3}^*(\mathbf{r}_3) u_{-\kappa', \ell_4}(\mathbf{r}_4)\tag{A13}$$

$$\begin{aligned}&\times e^{-(\mathbf{G}_1 - \mathbf{G}_2) \cdot \mathbf{r}_1} e^{(\mathbf{G}_4 - \mathbf{G}_3) \cdot \mathbf{r}_4} e^{\mathbf{G}_1 \cdot \mathbf{r}_1 - \mathbf{G}_2 \cdot \mathbf{r}_2 + \mathbf{G}_3 \cdot \mathbf{r}_3 - \mathbf{G}_4 \cdot \mathbf{r}_4} \\ &\times c_{\kappa, \ell_1, \sigma}^\dagger c_{\kappa', \ell_2, \sigma} c_{-\kappa, \ell_3, -\sigma}^\dagger c_{-\kappa', \ell_4, -\sigma}\end{aligned}\tag{A14}$$

$$\begin{aligned}&= \frac{1}{2q^2} \sum_{\substack{\kappa, \kappa', \sigma \\ \ell_1 \dots \ell_4}} \tilde{V}(\kappa - \kappa') \sum_{\substack{\mathbf{G}_1 \dots \mathbf{G}_3 \\ \mathbf{r}_1 \dots \mathbf{r}_4}} u_{\kappa, \ell_1}^*(\mathbf{r}_1) u_{\kappa', \ell_2}(\mathbf{r}_2) u_{-\kappa, \ell_3}^*(\mathbf{r}_3) u_{-\kappa', \ell_4}(\mathbf{r}_4) \\ &\quad \times e^{\mathbf{G}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)} e^{\mathbf{G}_3 \cdot (\mathbf{r}_3 - \mathbf{r}_4)} c_{\kappa, \ell_1, \sigma}^\dagger c_{\kappa', \ell_2, \sigma} c_{-\kappa, \ell_3, -\sigma}^\dagger c_{-\kappa', \ell_4, -\sigma} \\ &= \frac{1}{2} \sum_{\substack{\kappa, \kappa', \sigma \\ \ell_1 \dots \ell_4}} \tilde{V}(\kappa - \kappa') \sum_{\mathbf{r} \mathbf{r}'} u_{\kappa, \ell_1}^*(\mathbf{r}) u_{\kappa', \ell_2}(\mathbf{r}) u_{-\kappa, \ell_3}^*(\mathbf{r}') u_{-\kappa', \ell_4}(\mathbf{r}') c_{\kappa, \ell_1, \sigma}^\dagger c_{\kappa', \ell_2, \sigma} c_{-\kappa, \ell_3, -\sigma}^\dagger c_{-\kappa', \ell_4, -\sigma} \\ &= \frac{1}{2} \sum_{\substack{\kappa, \kappa', \sigma \\ \ell_1 \dots \ell_4}} \tilde{V}(\kappa - \kappa') \underbrace{\sum_{\mathbf{G}, \mathbf{G}'} \tilde{u}_{\kappa, \ell_1}^*(\mathbf{G}) \tilde{u}_{\kappa', \ell_2}(\mathbf{G}) \tilde{u}_{-\kappa, \ell_3}^*(\mathbf{G}') \tilde{u}_{-\kappa', \ell_4}(\mathbf{G}')}_{\equiv V_{\kappa, \kappa'}^{\ell_1 \ell_3, \ell_2 \ell_4}} c_{\kappa, \ell_1, \sigma}^\dagger c_{\kappa', \ell_2, \sigma} c_{-\kappa, \ell_3, -\sigma}^\dagger c_{-\kappa', \ell_4, -\sigma}.\end{aligned}\tag{A15}$$

From Ref. 3, we know that the term  $\tilde{V}(\kappa - \kappa')$  can be written as

$$\tilde{V}(\kappa - \kappa') = \sum_{\eta=1}^5 \lambda_\eta B_\eta(\kappa) B_\eta(\kappa'),\tag{A16}$$

where  $\boldsymbol{\lambda} = (U, V, W, V, W)$ , and

$$\begin{aligned} B_1(\boldsymbol{\kappa}) &= \sqrt{N} \\ B_2(\boldsymbol{\kappa}) &= \sqrt{N} [\cos(\kappa_x) + \cos(\kappa_y)] \\ B_3(\boldsymbol{\kappa}) &= \sqrt{2N} \cos(\kappa_x) \cos(\kappa_y) \\ B_4(\boldsymbol{\kappa}) &= \sqrt{N} [\cos(\kappa_x) - \cos(\kappa_y)] \\ B_5(\boldsymbol{\kappa}) &= \sqrt{2N} \sin(\kappa_x) \sin(\kappa_y). \end{aligned} \quad (\text{A17})$$

The upper limit  $\eta \leq 5$  in the sum is due to the even symmetry of the space wave-function.<sup>32</sup> We can then write

$$V_{\boldsymbol{\kappa}, \boldsymbol{\kappa}'}^{\ell_1 \ell_2, \ell'_1 \ell'_2} = \sum_{\boldsymbol{G}, \boldsymbol{G}'} \sum_{\eta=1}^5 \lambda_{\eta} \underbrace{(\tilde{u}_{\boldsymbol{\kappa}, \ell_1}^*(\boldsymbol{G}) \tilde{u}_{-\boldsymbol{\kappa}, \ell_2}^*(\boldsymbol{G}') B_{\eta}(\boldsymbol{\kappa}))}_{\equiv \tilde{B}_{\eta, \boldsymbol{G}, \boldsymbol{G}'}^*(\ell_1, \ell_2, \boldsymbol{\kappa})} \underbrace{(\tilde{u}_{\boldsymbol{\kappa}', \ell'_1}(\boldsymbol{G}) \tilde{u}_{-\boldsymbol{\kappa}', \ell'_2}(\boldsymbol{G}') B_{\eta}(\boldsymbol{\kappa}'))}_{\equiv \tilde{B}_{\eta, \boldsymbol{G}, \boldsymbol{G}'}(\ell'_1, \ell'_2, \boldsymbol{\kappa}')}. \quad (\text{A18})$$

To simplify notation it is useful to introduce the shorthand notations  $\xi = \{\eta, \boldsymbol{G}, \boldsymbol{G}'\}$ , and  $\chi = \{\ell_1, \ell_2, \boldsymbol{\kappa}\}$  which allows us to write

$$V_{\chi, \chi'} = \sum_{\xi} \lambda_{\xi} \tilde{B}_{\xi}^*(\chi) \tilde{B}_{\xi}(\chi'), \quad (\text{A19})$$

where  $\lambda_{\xi} \equiv \lambda_{\eta}$ .

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  - <sup>29</sup> E.g. a lattice constant  $a = 4\text{\AA}$  and  $B = 100\text{T}$  gives  $q \sim 250$  or  $q^2 \sim 10^5$ .
  - <sup>30</sup> In an external magnetic field  $|\epsilon(-\sigma) - \epsilon(\sigma)| = g_s \frac{e\hbar}{2m} B = 2\pi g_s^* \frac{B}{q}$  where  $m^*$  is the effective mass and  $g_s = 2$  in most circumstances. This is found by using that  $\frac{\hbar^2}{2m^* a^2} \equiv t$ ,  $\frac{2\pi B a^2}{\hbar/e} = 2\pi \frac{\phi}{\phi_0}$  and  $g_s \frac{m^*}{m} \equiv g^*$ .
  - <sup>31</sup> We can introduce “Wannier functions” or localized states  $|i\rangle = \sum_{\boldsymbol{\kappa}, \ell} \psi_{\boldsymbol{\kappa}, \ell}^*(\boldsymbol{r}_i) |\boldsymbol{\kappa}, \ell\rangle$ .
  - <sup>32</sup> The reason for the number  $\eta \leq 5$  in the zero-field situation is that the gap must be symmetric for  $\boldsymbol{k} \rightarrow -\boldsymbol{k}$ . A magnetic field destroys time reversal symmetry, and one may have to include both even and odd symmetries, then  $\eta \leq 9$ .